

Arc diagrams, flip distances, and Hamiltonian triangulations*

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Abstract

We show that every triangulation (maximal planar graph) on $n \geq 6$ vertices can be flipped into a Hamiltonian triangulation using a sequence of less than $n/2$ combinatorial edge flips. The previously best upper bound uses 4-connectivity as a means to establish Hamiltonicity. But in general about $3n/5$ flips are necessary to reach a 4-connected triangulation. Our result improves the upper bound on the diameter of the flip graph of combinatorial triangulations on n vertices from $5.2n - 33.6$ to $5n - 23$. We also show that for every triangulation on n vertices there is a simultaneous flip of less than $2n/3$ edges to a 4-connected triangulation. The bound on the number of edges is tight, up to an additive constant. As another application we show that every planar graph on n vertices admits an arc diagram with less than $n/2$ biarcs, that is, after subdividing less than $n/2$ (of potentially $3n - 6$) edges the resulting graph admits a 2-page book embedding.

1 Introduction

An *arc diagram* (Figure 1) is a drawing of a graph in which vertices are represented by points on a horizontal line, called the *spine*, and edges are drawn either as one halfcircle (*proper arc*) or as a sequence of halfcircles centered on the line (forming a smooth Jordan arc). In a *proper arc diagram* all arcs are proper. Arc diagrams have been used and studied in many contexts since their first appearance in the mid-sixties [28, 33]. They constitute a well-studied geometric representation in graph drawing [20] that occurs, for instance, in the study of crossing numbers [1, 6] and universal point sets for circular arc drawings [3].

Bernhart and Kainen [4] proved that a planar graph admits a *plane* (i.e., crossing-free) proper arc diagram if and only if it can be augmented to a Hamiltonian planar graph by adding new edges. Such planar graphs are also called *subhamiltonian*, and they are NP-hard to recognize [37]. A Hamiltonian cycle in the augmented graph directly yields a feasible order for

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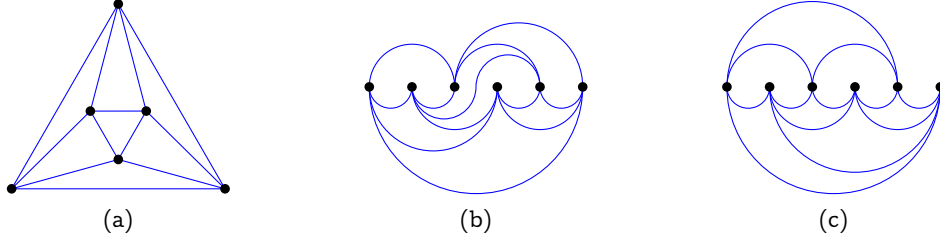


Figure 1: A plane straight-line drawing (a), an arc diagram (b) and a proper arc-diagram (c) of the same graph.

the vertices on the spine. Every planar graph can be subdivided into a subhamiltonian graph with at most one subdivision vertex per edge [29]. Consequently, every planar graph admits a plane *biarc diagram* in which each edge is either a proper arc or the union of two halfcircles (a *biarc*); one above and one below the spine. Di Giacomo et al. [21] showed that every planar graph even admits a *monotone* plane biarc diagram in which every biarc is x -monotone—such an embedding is also called a *2-page topological book embedding*. See [20] for various other applications of subhamiltonian subdivisions of planar graphs.

Eppstein [17] said: “Arc diagrams (with one arc per edge) are very usable and practical but can only handle a subset of planar graphs.” Using biarcs allows us to represent all planar graphs, but adds to the complexity of the drawing. Hence it is a natural question to ask: How close can we get to a proper arc diagram, while still being able to represent all planar graphs? A natural measure of complexity is the number of biarcs used.

Previous methods for subdividing an n -vertex planar graph into a subhamiltonian graph use at most one subdivision per edge [20, 21, 25, 29], consequently the number of biarcs in an arc diagram is bounded by the number of edges. Our main goal in this paper is to tighten the upper and lower bounds on the minimum number of biarcs in an arc diagram (or, alternatively, the number of subdivision vertices in a subhamiltonian subdivision) of a planar graph with n vertices. Minimizing the number of biarcs is clearly NP-hard, since the number of biarcs is zero if and only if the graph is subhamiltonian.

Our results. In Section 3 we show that the number of biarcs can be bounded by n , even when they are restricted to be monotone. Although previous methods can be shown to yield less than the trivial $3n - 6$ biarcs [25], or ensure monotonicity [21], we give the first proof that both properties can be guaranteed simultaneously. The algorithm is similar to the canonical ordering-based method of Di Giacomo et al. [21].

Theorem 1. *Every planar graph on $n \geq 4$ vertices admits a plane biarc diagram using at most $n - 4$ biarcs, all of which are monotone. Moreover, such a diagram can be computed in $O(n)$ time.*

For arbitrary (not necessarily monotone) biarcs we achieve better bounds. Our main tool is relating subhamiltonian planar graphs to edge flips and subdivisions in triangulations.

A *flip* in a triangulation involves switching the diagonal of a quadrilateral made of two adjacent facial triangles. We consider *combinatorial flips*, which can be regarded as an operation on an abstract graph. The *flip graph* induced by flips on the set of all triangulations on n vertices, and the corresponding *flip distance* between two triangulations, have been the topic of extensive research [9, 11]. For instance, the flip diameter restricted to the interior of a convex polygon is equivalent to the rotation distance of binary trees [31, 34].

By *subdividing* an edge e we mean replacing e with a new vertex that is connected to both

endpoints of e . The following theorem, which is proved in Section 4, relates biarcs to edge subdivisions and is a simple generalization of the characterization of Bernhart and Kainen.

Theorem 2. *A planar graph G admits a plane biarc diagram with at most k biarcs if and only if there is a set of at most k edges in G so that subdividing these edges transforms G into a subhamiltonian graph.*

In Section 5 we prove that in every triangulation there exists a set of less than $2n/3$ edges that can be flipped *simultaneously* so that the resulting triangulation is 4-connected, and that this bound is tight up to an additive constant. Since by Tutte's Theorem every 4-connected planar graph is Hamiltonian, we can transform every planar graph into a subhamiltonian graph by subdividing at most $2n/3$ edges. The fact that a single simultaneous flip can make a triangulation 4-connected has already been established by Bose et al. [8]. However, they do not give any bound on the number of flipped edges.

Theorem 3. *Every maximal planar graph on $n \geq 6$ vertices can be transformed into a 4-connected maximal planar graph using a simultaneous flip of at most $\lfloor (2n-7)/3 \rfloor$ edges. Moreover, such a set of simultaneously flippable edges can be computed in $O(n^2)$ time.*

Theorem 4. *For every $i \in \mathbb{N}$, there is a maximal planar graph G_i on $n_i = 3i + 4$ vertices such that no simultaneous flip of less than $(2n_i - 8)/3 = 2i$ edges results in a 4-connected graph.*

Finally, in Section 6 we prove an upper bound on the flip distance of a triangulation to Hamiltonicity, that is, on the worst-case number of successive flips required to reach a Hamiltonian triangulation. Given the hardness of determining whether a given planar graph is Hamiltonian, we should not expect a nice characterization of (non-)Hamiltonicity. Hence, in the context of planar graphs, 4-connectivity is often used as a substitute because by Tutte's Theorem it is a sufficient condition for Hamiltonicity.

Bose et al. [10] gave a tight bound (up to an additive constant) of $3n/5$ on the number of flips that transform a given triangulation on n vertices into a 4-connected triangulation. We show that fewer flips are sufficient to guarantee Hamiltonicity. Obviously, the target triangulation is not 4-connected in general, which means it possibly contains separating triangles.

Theorem 5. *Every maximal planar graph on $n \geq 6$ vertices can be transformed into a Hamiltonian maximal planar graph using a sequence of at most $\lfloor (n-3)/2 \rfloor$ edge flips. Alternatively, it can be transformed into a subhamiltonian planar graph by subdividing a set of at most $\lfloor (n-3)/2 \rfloor$ edges. Moreover, such a sequence of flips or subdivisions can be computed in $O(n^2)$ time.*

In this case we do not have a matching lower bound. The best lower bound we know can be obtained using *Kleetopes* [22]. These are convex polytopes that are generated from another convex polytope by replacing every face by a small pyramid. In the language of planar graphs, we start from a 3-connected planar graph and for every face add a new vertex that is connected to all vertices on the boundary of the face. If the graph we start from has enough faces, then the added vertices form a large independent set so that the resulting graph is not Hamiltonian. Aichholzer et al. [2] describe such a construction explicitly in the context of flipping a triangulation to a Hamiltonian triangulation, but state the asymptotics only. A precise counting reveals the following figures.

Theorem 6. *For every $i \in \mathbb{N}$, there is a maximal planar graph G_i on $n_i = 3i + 8$ vertices such that no sequence of less than $(n_i - 8)/3 = i$ edge flips produces a Hamiltonian graph, and there is no set of less than $(n_i - 8)/3 = i$ edges whose subdivision produces a subhamiltonian graph.*

Our proof of Theorem 5 is constructive, and each flip in the sequence involves an edge of the initial graph G that is incident to a separating triangle of G . Several of these edges may be incident to a common facial triangle, in which case the edges are not simultaneously flippable.

Theorem 2 allows us to translate Theorems 5 and 6 to the context of biarc diagrams, where we obtain bounds for the number of biarcs needed.

Corollary 7. *Every planar graph on $n \geq 6$ vertices admits a plane biarc diagram with at most $\lfloor (n-3)/2 \rfloor$ biarcs. Moreover, such a diagram can be computed in $O(n^2)$ time.*

Corollary 8. *For every $i \in \mathbb{N}$, there is a maximal planar graph G_i on $n_i = 3i + 8$ vertices that cannot be drawn as a plane biarc diagram using less than $(n_i - 8)/3 = i$ biarcs.*

As another corollary, we establish a new upper bound on the diameter of the flip graph of all triangulations on n vertices, improving on the previous best bound of $5.2n - 33.6$ by Bose et al. [10]. Mori et al. [27] showed that any two Hamiltonian triangulations on n vertices can be transformed into each other by a sequence of at most $\max\{4n - 20, 0\}$ flips. Combined with Theorem 5, this implies the following.

Corollary 9. *Every two triangulations on $n \geq 6$ vertices can be transformed into each other using a sequence of at most $5n - 23$ edge flips.*

2 Notation

A *drawing* of a graph G in \mathbb{R}^2 maps the vertices into distinct points in the plane and maps each edge to a Jordan arc between (the images of) the two vertices that is disjoint from (the image of) any other vertex. To avoid notational clutter it is common to identify vertices and edges with their geometric representation. A drawing is called *plane* (or an *embedding*) if no two edges intersect except at a possible common endpoint. Only planar graphs admit plane drawings, but not every drawing of a planar graph is plane. A *maximal planar* graph on n vertices is a planar graph with $3n - 6$ edges. In this paper the term *triangulation* is used as a synonym for maximal planar graph.¹

In a plane drawing of a triangulation G , every face (including the outer face) is bounded by three edges. Hence, every triangulation with $n \geq 4$ vertices is 3-connected [16][Lemma 4.4.5]. Every 3-connected planar graph has a topologically unique plane drawing, apart from the choice of the outer face. Specifically, the facial triangles are precisely the nonseparating chordless cycles of G in every plane drawing [16][Proposition 4.2.7]. Consequently, G has a well-defined dual graph G^* (independent of the drawing): the vertices of G^* correspond to the faces of G , and two vertices of G^* are adjacent if and only if the corresponding faces share an edge. A triangle of G that is not facial is called a *separating* triangle, as its removal disconnects the graph.

A graph is *Hamiltonian* if it contains a cycle through all vertices. By a famous theorem of Tutte [35, 36], all 4-connected planar graphs are Hamiltonian. For triangulations, 4-connectivity is equivalent to the absence of separating triangles. A vertex or an edge is *incident* to a triangle T in a graph if it is a vertex or edge of T .

A triangulation G can be partitioned into a *4-block tree* \mathcal{B} . Each vertex of \mathcal{B} is either a maximal 4-connected component of G or a subgraph of G that is isomorphic to K_4 . Two vertices of \mathcal{B} are adjacent if they share a separating triangle of G . The 4-block tree is similar to the standard block-tree for 2-connected components, but the generalization of the notion

¹In contrast, a maximal plane *straight-line* drawing may have fewer edges, depending on the number of points on the convex hull.

“component” to higher connectivity is not straightforward in general. For a triangulation, however, the 4-block tree is well-defined and can be computed in linear time and space [24].

Flips. Consider an edge ab of a triangulation G and let abc and adb denote the two incident facial triangles. The *flip* of ab replaces the edge ab by the edge cd . If this operation produces a triangulation (i.e., if $c \neq d$ and the edge cd is not already present in G), we call ab *flippable*².

A closely related concept is the *simultaneous flip* of a set F of flippable edges in a triangulation $G = (V, E)$, which is defined as follows. For $e \in F$ denote by $c(e)$ the edge created by flipping e in G , and let $C(F) = \bigcup_{e \in F} c(e)$. Then the simultaneous flip of F in G results in the graph $G' = (V, (E \setminus F) \cup C(F))$. Bose et al. [8] introduced this notion and showed that the result of a simultaneous flip is a triangulation if every facial triangle of G is incident to at most one edge from F and the edges $c(e)$, for $e \in F$, are all distinct and not present in E .

3 Monotone Biarc Diagrams

In this section we present a simple linear time algorithm to construct a biarc diagram in which all biarcs are drawn as monotone curves (with respect to the spine). The algorithm is based on the fundamental notion of a canonical ordering, which is defined for an *embedded* triangulation. As every triangulation on $n \geq 4$ vertices is 3-connected, embedding it into the plane essentially amounts to selecting one facial triangle to be the *outer face*. This choice also determines a unique outer face (cycle) for every biconnected subgraph.

A *canonical ordering* [19] for an embedded triangulation G on n vertices is a total order of the vertices v_1, \dots, v_n such that

- for $i \in \{3, \dots, n\}$, the induced subgraph $G_i = G[\{v_1, \dots, v_i\}]$ is biconnected and internally triangulated (i.e., every face other than the outer face C_i is a triangle);
- for $i \in \{3, \dots, n\}$, $v_1 v_2$ is an edge of C_i ;
- for $i \in \{3, \dots, n-1\}$, v_{i+1} lies in the interior of C_i (the unbounded region of the plane bounded by C_i) and the neighbors of v_{i+1} in G_i form a sequence of consecutive vertices along the boundary of C_i .

It is well-known that every triangulation admits a canonical ordering [19], and such an ordering can be computed in $O(n)$ time [15].

Theorem 1. *Every planar graph on $n \geq 4$ vertices admits a plane biarc diagram using at most $n-4$ biarcs, all of which are monotone. Moreover, such a diagram can be computed in $O(n)$ time.*

Proof. Let G be a planar graph on $n \geq 4$ vertices and suppose without loss of generality that G is an embedded triangulation. If G is not maximal planar, add edges to make G maximal planar, choose any embedding, and simply remove the added edges from the final drawing.

Consider a canonical ordering v_1, \dots, v_n of the vertices of G . We construct a biarc diagram of G incrementally by inserting the vertices in canonical order and embedding them on the x -axis (spine). Let $G_i = G[\{v_1, \dots, v_i\}]$ and let C_i denote the outer cycle of G_i , for $i = 3, \dots, n$. During the algorithm, we maintain the following invariants.

- (I1) All edges of C_i are proper arcs (none is a biarc). The vertices v_1 and v_2 are the leftmost and rightmost, respectively, vertices of G_i on the spine. The edge $v_1 v_2$ forms the lower

²We consider *combinatorial flips*, as opposed to *geometric flips* defined for straight-line plane drawings, where an edge is flippable if and only if the quadrilateral formed by the two incident facial triangles is convex.

envelope of G_i (i.e., no point of the biarc diagram is vertically below). All edges of C_i other than v_1v_2 are on the upper envelope of G_i (i.e., no point of the biarc diagram is vertically above).

- (I2) Any biarc used in G_i is a down-up biarc, that is, the semicircle incident to its left endpoint lies below the spine and the semicircle incident to its right endpoint lies above the spine.

We embed the triangle G_3 by placing v_1 , v_3 , and v_2 on the spine in this order from left to right and by drawing all edges as proper arcs below the spine (Figure 2). Clearly (I1)–(I2) hold for this embedding.

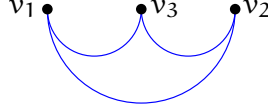


Figure 2: Start the incremental embedding with a triangle.

Now suppose that we have a biarc diagram for G_i that satisfies the invariants and we want to add v_{i+1} . Let w_1, \dots, w_{k_i} be the vertices of C_i labeled from left to right along the spine. By (I1) this order is compatible with the vertex order along C_i , with $v_1 = w_1$ and $v_2 = w_{k_i}$. As we work with a canonical ordering, the neighbors of v_{i+1} on C_i form a contiguous subsequence $w_{\ell_i}, \dots, w_{r_i}$ of C_i , with $1 \leq \ell_i < r_i \leq k_i$. In addition, (I2) guarantees that we can insert v_{i+1} along the spine between w_{ℓ_i} and w_{ℓ_i+1} , just to the right of w_{ℓ_i} : Every biarc leaving w_{ℓ_i} to the right goes down first and, therefore, does not block the spine locally at w_{ℓ_i} , whereas proper arcs above the spine leaving w_{ℓ_i} to the right can be bent down to become down-up biarcs while maintaining their vertical order (Figure 3). After placing v_{i+1} , the edges to $w_{\ell_i}, \dots, w_{r_i}$ can be drawn as proper arcs above the spine. The edge $w_{\ell_i}v_{i+1}$ can even be drawn as a proper arc below the spine because the two vertices are neighbors along the spine by construction. It is easily checked that the invariants (I1)–(I2) are maintained. This completes the description of the first version of our algorithm.

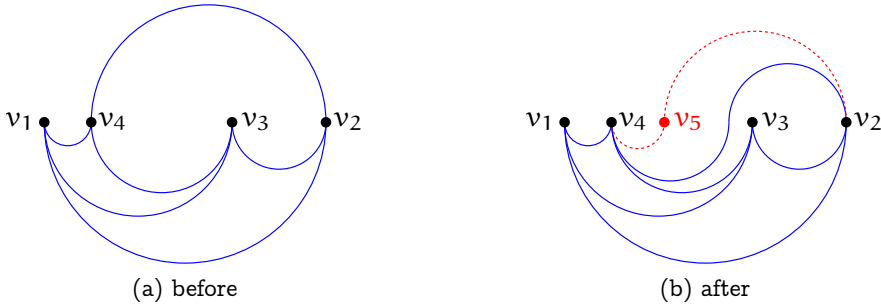


Figure 3: Make room for a new vertex v_5 .

First lower bound on the number of proper arcs. It remains to bound the number of biarcs used by the algorithm. As a first observation, note that all edges are drawn as proper arcs initially (when they first appear). An edge e may become a biarc in a later step only if it is bent down to make room for a vertex inserted immediately to the right of the left endpoint of e . In particular, every edge drawn below the spine, such as the three edges of G_3 and the edges $w_{\ell_i}v_{i+1}$ drawn at steps $i \in \{3, \dots, n-2\}$, remain proper arcs throughout the algorithm. Finally, at least three new edges will be drawn in the last step $i = n-1$ (i.e., when inserting v_n) as proper arcs. This

yields a first lower bound of at least $3 + (n - 4) + 3 = n + 2$ proper arcs and, therefore, at most $3n - 6 - (n + 2) = 2n - 8$ biarcs.

A refined algorithm and lower bound. In order to obtain the claimed bound, let us consider in more detail the insertion of a vertex v_{i+1} where $i \in \{3, \dots, n-1\}$. We claim that for any vertex v_{i+1} we can obtain $r_i - \ell_i$ proper arcs in the final drawing, rather than just one. However, we also have to adapt our algorithm slightly, as described in the following paragraph.

The improvement is based on two simple but crucial observations. First, observe that a vertex v_{i+1} can be inserted just to the right of any of the vertices $w_{\ell_i}, \dots, w_{r_i-1}$, not only w_{ℓ_i} . The invariants (I1)–(I2) can be maintained for any such choice. Second, observe that none of the edges of the path $w_{\ell_i+1}, \dots, w_{r_i}$ appear on C_{i+1} anymore and neither do the left endpoints of these edges. In particular, it follows that every proper arc among those edges will remain a proper arc in the final drawing. Now we have to be careful when counting these edges because some of them might be drawn below the spine and we accounted for them already. Here is where the first observation comes to our help. We modify the algorithm to insert v_{i+1} just to the right of the last vertex w_{f_i} in $w_{\ell_i}, \dots, w_{r_i-1}$ such that the edge $w_{f_i}w_{f_i+1}$ is drawn below the spine. If no such edge exists, then we insert v_{i+1} just to the right of w_{ℓ_i} , as before.

For the analysis we consider two cases. If w_{f_i} exists (Figure 4), then the insertion of v_{i+1} does not create any biarcs. All edges along the path $w_{f_i+1}, \dots, w_{r_i}$ are proper arcs drawn above the spine and have not yet been counted. As none of these arcs appears on C_{i+1} , they will not be counted again. In addition, all edges from v_{i+1} to $w_{\ell_i+1}, \dots, w_{f_i}$ are proper arcs of G_{i+1} whose left endpoints do not appear on C_{i+1} . Lastly, the edge from v_{i+1} to w_{f_i+1} can be drawn below the spine. Therefore, all these edges remain proper arcs throughout the algorithm. The total number of new proper arcs in the final drawing is, therefore, at least $(r_i - f_i - 1) + (f_i - \ell_i) + 1 = r_i - \ell_i$.

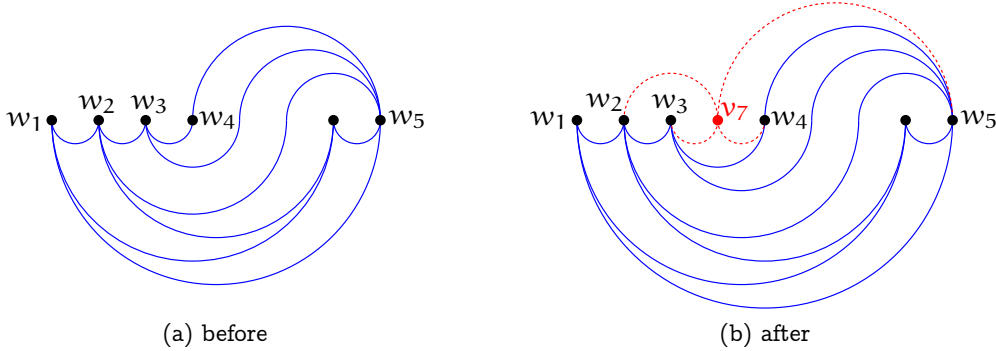


Figure 4: Inserting a new vertex v_7 with $\ell_6 = 2$, $r_6 = 5$, and $f_6 = 3$.

In the second case there is no w_{f_i} and v_{i+1} is inserted just to the right of w_{ℓ_i} . But we also know that none of the edges in the path $w_{\ell_i+1}, \dots, w_{r_i}$ of C_i are below the spine. Therefore, all these edges will be proper arcs in the final drawing that have not been counted yet. Together with the new edge $v_{i+1}w_{\ell_i}$, which is drawn below the spine, we get again $r_i - \ell_i$ new proper arcs in the final drawing.

In summary, we *always* get at least $r_i - \ell_i$ new proper arcs in the final drawing when inserting a vertex v_{i+1} . In the last step, when inserting v_n , we even get $r_{n-1} - \ell_{n-1} + 1$ new proper arcs. Therefore, the total number of proper arcs is bounded from below by $4 + \sum_{i=3}^{n-1} (r_i - \ell_i)$. The

total number of edges in G is

$$3n - 6 = 3 + \sum_{i=3}^{n-1} (r_i - \ell_i + 1).$$

Combining both expressions yields at least

$$4 + \sum_{i=3}^{n-1} (r_i - \ell_i) = 4 + 2n - 6 = 2n - 2$$

proper arcs and, therefore, at most $3n - 6 - (2n - 2) = n - 4$ biarcs in the final drawing.

Regarding the runtime bound, observe that when inserting a new vertex we inspect all its neighbors on the current outer cycle to select the right spot for insertion. Therefore the time spent for each vertex is proportional to its degree. As the graph is planar, the sum of all vertex degrees is linear. The arc diagram under construction can be represented as a tree using standard techniques [15], where in addition we also store for every edge whether it is a proper arc or a down-up biarc. \square

4 General Biarc Diagrams

In this section we discuss the connection between biarc diagrams and edge flips and subdivisions in triangulations. Recall that Bernhart and Kainen [4] characterized planar graphs that admit a plane proper arc diagram as all subhamiltonian planar graphs. The following theorem generalizes this characterization in the context of biarc diagrams (the original Theorem is obtained by setting $k = 0$).

Theorem 2. *A planar graph G admits a plane biarc diagram with at most k biarcs if and only if there is a set of at most k edges in G so that subdividing these edges transforms G into a subhamiltonian graph.*

Proof. First, suppose there is a biarc diagram of G with at most k biarcs. Then we can simply subdivide these at most k biarcs in order to obtain a proper arc diagram of some graph G' . By the characterization of Bernhart and Kainen, G' is subhamiltonian.

Second, fix a set of at most k edges in G so that subdividing them results in a subhamiltonian graph G' . By the characterization of Bernhart and Kainen we know that G' admits a proper arc diagram. Removing the new vertices from the subdivided edges in that arc diagram results in a biarc diagram of G with at most k biarcs (if both arcs incident to a subdivision vertex are on the same side of the spine, then the biarc can be replaced by a single proper arc). \square

A similar statement can be obtained for simultaneous edge flips, where the edges to be manipulated must not share a triangle. As this is a more restricted setting, we get a correspondence in one direction only. But this is enough for the purpose of getting upper bounds on the number of biarcs.

Lemma 10. *If a maximal planar graph G can be transformed into a Hamiltonian graph with a simultaneous flip of k edges, then G admits a plane biarc diagram with at most k biarcs.*

Proof. Let H be a Hamiltonian graph obtained from G by simultaneously flipping an edge set E_1 to E_2 with $|E_1| = k$. Without loss of generality, assume that E_1 is a minimal set of edges that must be flipped in order to obtain a Hamiltonian graph. Consequently, every Hamiltonian cycle in H passes through all k edges in E_2 . If we subdivide each edge in E_2 , we obtain another

Hamiltonian graph H' . Now consider the graph G' obtained from G by subdividing each edge in E_1 , and identify the subdivision vertices of the corresponding edges in G' and H' . Notice that the union of G' and H' is a plane graph that contains H' , hence it is Hamiltonian. Consequently G' is subhamiltonian. By Theorem 2, G admits a plane biarc diagram with at most k biarcs, as claimed. \square

In order to obtain a general statement about arc diagrams from Lemma 10, we need a bound on the number of edges to simultaneously flip in a given graph in order to make it Hamiltonian. Even the existence of such a simultaneous flip—regardless of the number of edges involved—is not obvious to begin with. For instance, consider triangulations G_1 and G_2 where G_1 has a vertex with linear degree and all vertices in G_2 have constant degree (e.g., a nested triangle graph). As a single simultaneous flip can only change about half of the edges incident to a vertex, at least a logarithmic number of simultaneous flips is required to transform G_1 into G_2 [8].

Bose et al. [8] showed that every triangulation on $n \geq 6$ vertices can be transformed to a 4-connected (hence Hamiltonian) triangulation by a single simultaneous flip. However, no bound is known on the number of flipped edges, which leaves us with the trivial bound of $(2n - 4)/2 = n - 2$. Note that the corresponding bound on the number of biarcs is similar to the one from Theorem 1, but there we could guarantee that all biarcs are monotone. Using Lemma 10 we do not have any control over the type of biarcs used.

5 Simultaneous Flip Distance to 4-connectivity

In this section we determine the maximum number of edges needed to transform an n -vertex triangulation into a 4-connected triangulation using a single simultaneous flip. Consider a triangulation $G = (V, E)$. As there is no 4-connected triangulation on fewer than six vertices, suppose that G has at least six vertices. We would like to transform G into a 4-connected triangulation by simultaneously flipping a set $F \subseteq E$ of edges such that all separating triangles are destroyed and none created. We use the following criterion to ensure that the resulting triangulation is 4-connected.

Lemma 11 (Bose et al. [8]). *Let F be a set of edges in a triangulation G such that no two edges in F are incident to a common triangle, every edge in F is incident to a separating triangle, and for every separating triangle T there is at least one edge in F that is incident to T . Then F is simultaneously flippable in G and the resulting triangulation is 4-connected.*

Recall that the edges of a triangulation G and its dual G^* are in one-to-one correspondence. Consequently, the set F^* of edges dual to those in F forms a matching in G^* . As all faces of a triangulation are triangles, G^* is cubic (3-regular). Moreover, every triangulation on $n \geq 4$ vertices is 3-connected and so its dual is bridgeless (2-edge-connected). By a famous theorem of Tait the following statement is equivalent to the Four-Color Theorem:

Theorem 12 (Tait [7]). *Every bridgeless cubic planar graph admits a partition of the edge set into three perfect matchings.*

In particular, this applies to the dual of a triangulation. Call a set $F \subseteq E$ of edges of a triangulation $G = (V, E)$ a (perfect) *dual matching* if F^* forms a (perfect) matching of G^* . While it is clear that a perfect dual matching contains exactly one edge of each facial triangle,

this is not obvious for separating triangles. But it follows from a simple parity argument, as the following lemma shows.³

Lemma 13. *Every perfect dual matching of a triangulation G contains an edge of every triangle of G .*

Proof. For facial triangles the statement holds by definition. So consider a separating triangle T of G and the subgraphs H and H' of G induced by T together with the two respective components of $G \setminus T$. As H is a maximal planar graph, it has $2|V(H)| - 4$ faces including the facial triangle T . Hence the number of faces of H different from T is odd and so every perfect matching of G^* contains at least one edge that connects a face of H with a face of H' . The corresponding primal edge of the dual matching is an edge of T , as required. \square

The combination of Theorem 12 with Lemma 13 immediately yields the following

Corollary 14. *Every triangulation G admits a partition of the edge set into three perfect dual matchings such that every triangle of G is incident to exactly one edge from each of the three matchings.*

The last missing bit to prove Theorem 3 is an upper bound on the number of edges in a triangulation that can be incident to separating triangles.

Lemma 15. *At most $2n - 7$ edges of a maximal planar graph on $n \geq 4$ vertices are incident to separating triangles. This bound is the best possible.*

Proof. We proceed by induction on the number of separating triangles. For a maximal planar graph without separating triangles the statement is trivial. For $n = 4$, the only maximal planar graph is K_4 and it has no separating triangle. For $n = 5$, there is only one maximal planar graph up to isomorphism, and it contains exactly one separating triangle, bounded by $3 = 2 \cdot 5 - 7$ edges.

Consider a maximal planar graph G on $n \geq 6$ vertices and a minimal separating triangle T of G , that is, a separating triangle such that for at least one component C of $G \setminus T$ the subgraph $H := G[C \cup T]$ does not contain a separating triangle (equivalently, $H = K_4$ or H is 4-connected). Put $k = |C| \in \{1, \dots, n - 4\}$. The graph $G' = G \setminus C$ has $n - k$ vertices and contains exactly one fewer separating triangle than G . By the inductive hypothesis, at most $2(n - k) - 7$ edges of G' are incident to separating triangles of G' . As far as the corresponding count for G is concerned, only the three edges of T have to be accounted for in addition.

If some edge of T also bounds a separating triangle in G' , then this edge has already been counted inductively in G' . Including the remaining at most two edges of T , we see that at most $2(n - k) - 7 + 2 \leq 2n - 7$ edges of G are incident to separating triangles of G . Also if $k \geq 2$, then at most $2(n - k) - 7 + 3 \leq 2n - 8$ edges of G are incident to separating triangles of G .

Otherwise, $k = 1$ and none of the edges of T is incident to any separating triangle in G' . Denote the vertices of T by $T = (a, b, c)$, and let T and (b, a, d) be the two faces of G' incident to the edge ab . By contracting the edge ab in G' , we obtain a graph G'' on $n - 2$ vertices. The contraction identifies the two edges ac and bc into a single edge. Similarly the two edges ad and bd are identified into a single edge.

We claim that after this contraction G'' is simple, that is, no multi-edge is introduced (other than the two edge pairs already mentioned and handled). This is because the vertices a and b have exactly two common neighbors in G' , which are c and d . If a and b had any other common neighbor $w \notin \{c, d\}$, then the triangle abw would be a separating triangle in G' , contrary to

³Bose et al. [8] derive this property from the explicit Tait coloring. The statement here is slightly more general because it holds for every perfect dual matching.

our assumption that ab is not incident to any separating triangle in G' . Hence b and d are the only common neighbors of a and b , and so no multi-edge is created by contracting ab , as claimed.

Finally we observe that by the inductive hypothesis at most $2(n-2) - 7 = 2n - 11$ edges of G'' are incident to separating triangles of G'' . In addition to the three edges of T we also have to account for changes caused by the contraction of the edge ab . Edges ac and bc are identified in G'' , but neither is incident to any separating triangle in G' by assumption. Edges ad and bd are also identified. They each may be incident to separating triangles in G' but they are counted once only in G'' . Consequently, we count the three edges of T and one additional edge for a total of at most $(2n - 11) + 3 + 1 = 2n - 7$ edges incident to separating triangles in G .

For a matching lower bound, consider the graphs depicted in Figure 5. The solid edges are incident to separating triangles. On the left, we have $n = 6$ and $2n - 7 = 5$ and exactly 5 edges incident to separating triangles. To obtain larger examples, repeatedly insert a new vertex into a face with exactly one solid edge. The remaining two edges of this face become solid. Note that this operation creates a face with exactly one edge that is incident to a separating triangle, and so the operation can be repeated indefinitely. After k such operations we have $n = 6 + k$ vertices and precisely $5 + 2k = 2n - 7$ edges incident to separating triangles, as desired. \square

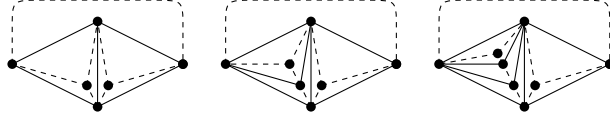


Figure 5: Tight examples for Lemma 15, for $n = 6, 7, 8$.

Now we have all pieces together to prove Theorem 3.

Theorem 3. *Every maximal planar graph on $n \geq 6$ vertices can be transformed into a 4-connected maximal planar graph using a simultaneous flip of at most $\lfloor (2n - 7)/3 \rfloor$ edges. Moreover, such a set of simultaneously flippable edges can be computed in $O(n^2)$ time.*

Proof. Consider a maximal planar graph G on n vertices. By Corollary 14 the $3n - 6$ edges of G can be partitioned into three perfect dual matchings D_1 , D_2 , and D_3 , of $n - 2$ edges each, such that each separating triangle is incident to one edge from each. Let M_i , for $i \in \{1, 2, 3\}$, denote the dual matching that results from removing all edges from D_i that are not incident to any separating triangle. By Lemma 15 at most $2n - 7$ edges of G are incident to separating triangles. Therefore, one of M_1 , M_2 , and M_3 contains at most $\lfloor (2n - 7)/3 \rfloor$ edges. By Lemma 11 these edges are simultaneously flippable and the resulting graph is 4-connected.

All separating triangles (and incident edges) can be found in $O(n)$ time [13]. Theorem 12 is known to be equivalent to the Four Color Theorem [7], and a proper 4-coloring of G yields an edge partition into dual matchings in all 4-connected subgraphs in $O(n)$ time. The current best algorithm for 4-coloring a planar graph with n vertices runs in $O(n^2)$ time [32]. Consequently, we can find a smallest dual matching from $\{M_1, M_2, M_3\}$ in $O(n^2)$ time. \square

The following construction shows that the bound in Theorem 3 is tight up to an additive constant of ± 1 .

Theorem 4. *For every $i \in \mathbb{N}$, there is a maximal planar graph G_i on $n_i = 3i + 4$ vertices such that no simultaneous flip of less than $(2n_i - 8)/3 = 2i$ edges results in a 4-connected graph.*

Proof. Start with $G_0 = K_4$ and select a face f_0 of G_0 . For $i \in \mathbb{N}$, the graph G_i is recursively obtained from G_{i-1} as follows (see Figure 6 where f_0 is the outer face): For each face f adjacent to f_0 in G_{i-1} , insert a new vertex of degree 3 into f and connect it to all three vertices of f . Since f_0 is adjacent to three distinct faces, the number of vertices in G_i is $n_i = 3i + 4$. By construction, G_i has three groups of separating triangles. Each group contains i separating triangles that lie in one of the three subdivided faces of G_0 and share a common edge with f_0 .

As the face f_0 is incident to all $3i$ separating triangles in G_i , it is tempting to just flip the three edges of f_0 . However, a simultaneous flip can include at most one of the edges incident to f_0 . Consequently, at least two edges of f_0 remain untouched, each of which is incident to a group of i separating triangles. As no two triangles within a group share any other edge, one flip per triangle is needed to destroy them all simultaneously. Also two separating triangles from different groups are edge-disjoint—except for the three largest separating triangles, which are bounded by the edges of G_0 . But any two groups share only one such edge and so at most one flip can be saved in this way. Therefore, in order to handle the two groups whose edge incident to f_0 is not flipped at least $2i - 1$ edges need to be flipped. Clearly, at least one more edge flip is required to handle the third group, which leaves us with the claimed bound of at least $2i$ edges. \square

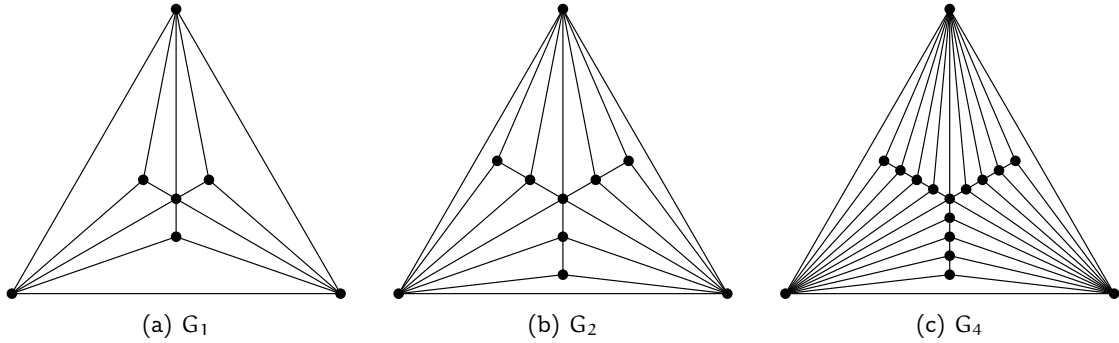


Figure 6: The first members of a family of triangulations that require a simultaneous flip of at least $(2n - 8)/3$ edges to become 4-connected.

6 Flip Distance to Hamiltonicity

With regard to arc diagrams, there is actually no reason to insist that the triangulation be 4-connected. In order to apply Lemma 10 we need only that the triangulation is Hamiltonian. In this section we go one step further and in addition lift the restriction that the flip be simultaneous. Instead, an arbitrary sequence of edge flips is allowed. In this case tight bounds are known if the goal is to obtain a 4-connected triangulation. Bose et al. [10] showed that $\lfloor (3n-9)/5 \rfloor$ flips are always sufficient and sometimes $(3n-10)/5$ flips are necessary to transform a given triangulation on n vertices into a 4-connected triangulation.

In general, a sequence of flips has no direct implication for arc diagrams. But if only edges of the original triangulation are flipped, then we can subdivide those edges rather than flipping them. In the resulting arc diagram only the subdivided edges may appear as biarcs. But a bound on the flip distance to a Hamiltonian triangulation is of independent interest. For instance, it is directly related to the current best upper bound on the diameter of the flip graph of combinatorial triangulations [10, 26, 27]. The argument uses a single so-called canonical triangulation

and shows that every triangulation can be transformed into this canonical triangulation in two steps: First at most $\lfloor (3n - 9)/5 \rfloor$ flips are needed to obtain a 4-connected triangulation and then an additional at most $2n - 15$ flips are needed to transform any 4-connected triangulation into the canonical one. Combining two such flip sequences yields an upper bound of $5.2n - 33.6$ on the diameter of the flip graph [10]. The bound of $2n - 15$ flips for the second step is actually tight [26]. The corresponding bound for a triangulation that is Hamiltonian (but not necessarily 4-connected) is slightly worse only: It can be transformed into the canonical triangulation using at most $2n - 10$ flips [27]. Hence our focus is to improve the first step by showing that fewer flips are needed to guarantee a Hamiltonian triangulation than a 4-connected one.

Theorem 5. *Every maximal planar graph on $n \geq 6$ vertices can be transformed into a Hamiltonian maximal planar graph using a sequence of at most $\lfloor (n - 3)/2 \rfloor$ edge flips. Alternatively, it can be transformed into a subhamiltonian planar graph by subdividing a set of at most $\lfloor (n - 3)/2 \rfloor$ edges. Moreover, such a sequence of flips or subdivisions can be computed in $O(n^2)$ time.*

Proof outline. The proof is constructive and consists of two steps. In a first step we apply a sequence of elementary operations that transform a triangulation G into a 4-connected triangulation G' . An elementary operation is either a usual edge flip or a *dummy flip*, where a facial triangle T is subdivided into three triangles by inserting a new (dummy) vertex and then all three edges of T are flipped. All this will be done in such a way that G' becomes 4-connected and, therefore, contains a Hamiltonian cycle H' . We then remove all dummy vertices and construct a Hamiltonian cycle H'' resembling H' in the resulting triangulation G'' . Finally, we argue that G'' can be obtained from G with at most $n/2$ (usual) edge flips. Specifically, we show that each dummy flip can be implemented using at most two edge flips.

Dummy flips. Given a triangulation G on $n \geq 4$ vertices and a facial triangle T of G , a dummy flip of T transforms G as follows (Figure 7): First, insert a new (dummy) vertex v in the interior of face T and connect it to all three vertices of T . Note that T becomes a separating triangle in the resulting graph. Second, flip all three edges of T in an arbitrary order. Similarly to the

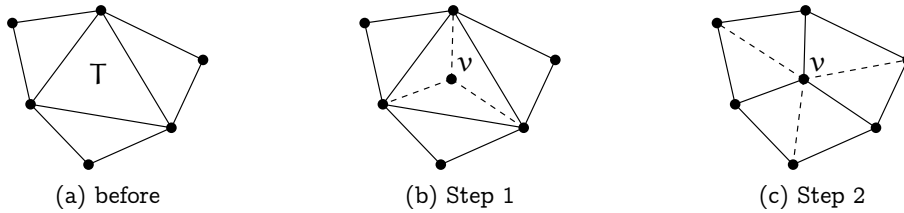


Figure 7: Example of a dummy flip.

usual flip operation, a dummy flip may create multiple edges. But we will use this operation in specific situations only—as specified in the lemma below—where we can show that it produces a triangulation (that is, no multiple edges).

Lemma 16. *Let G be a maximal planar graph and let T be a facial triangle of G such that every edge of T is incident to a separating triangle of G . Then the dummy flip operation of T in G produces no parallel edges and no new separating triangles.*

Proof. Let $T = abc$ be a facial triangle of G as specified above and insert a new vertex v into T . First we claim that every edge of T is flippable. Consider the edge ab and assume that it is incident to faces $T' = abv$ and $T'' = abd$. The only obstruction to flippability of ab is the

presence of an edge vd in G . By assumption there is a separating triangle $S = abe$ in G , for some vertex e . Given that both T' and T'' are facial, the vertices v and d are separated by S (they are in different components of $G \setminus S$). Therefore, by planarity of G , the edge vd is not present in G and so ab is flippable, as claimed.

Noting that any two distinct triangles in a (simple) graph share at most one edge, we observe that no separating triangle shares two edges with T . In particular, flipping the edge ab does not destroy any separating triangle incident to the edges bc or ca . Hence even after flipping one or two edges of T , we can still apply the above reasoning to show that the other edge(s) of T remain flippable. It follows that all three edges of T can be flipped in any order.

It remains to show that these flips do not introduce any separating triangle. Denote by G' the graph that results from the dummy flip of T in G . As all newly introduced edges are incident to v , any new separating triangle must also be incident to v . So suppose $S = vwx$ is a separating triangle in G' . In particular, this means that the edge wx was present in G already. At most one of w and x can be vertices of T , otherwise wx would not be an edge of G' (exactly the edges of T were flipped away, after all). So we may suppose without loss of generality that w is a vertex of some triangle $T' = abw$ in G . However, by assumption there is a separating triangle incident to the edge ab in G , which separates w (in G) from all neighbors of v in G' other than a and b . It follows that $x \in \{a, b\}$, but the triangles vwa and vwb are facial in G' by construction. Therefore, there is no separating triangle in G' that is incident to v and so no separating triangle is introduced by the dummy flip of T in G . \square

6.1 First Step: Establish 4-Connectedness

Our main lemma to establish Theorem 5 is the following.

Lemma 17. *Every maximal planar graph on $n \geq 6$ vertices can be transformed into a 4-connected maximal planar graph by a sequence of f flip and d dummy flip operations, for some $f, d \in \mathbb{N}$, such that $f + 2d \leq (n - 3)/2$.*

Recall that there are triangulations on n vertices that contain $\lfloor (3n - 9)/5 \rfloor$ pairwise edge-disjoint separating triangles [10, 23]. In this case, we need to flip away at least one edge from each separating triangle to reach 4-connectivity. Considering that a dummy flip operation flips three edges, we must have $f + 3d \geq \lfloor (3n - 9)/5 \rfloor$. The crucial claim in Lemma 17 is that $f + 2d \leq (n - 3)/2$ is possible, and later we will show how to replace each dummy flip by two usual flips rather than three (Lemma 29).

The rest of this section is devoted to the proof of Lemma 17. We describe an algorithm that, given a triangulation G on $n \geq 6$ vertices, returns a sequence of f flip and d dummy flip operations that produces a 4-connected graph. The bound is written equivalently as $6f + 12d \leq 3n - 9$ and is established via the following charging scheme. Each edge of G , with the exception of the three edges of the outer face, receives one unit of credit. Each edge flip costs six units. Each dummy flip costs fifteen units and produces three new edges, each of which receives one unit of credit.

4-Block Decomposition. In our algorithm, we recursively process 4-connected subgraphs using the 4-block tree \mathcal{B} of G (see Figure 8 for an example). By fixing an (arbitrary) plane embedding of G , we make \mathcal{B} a rooted tree such that the root is the 4-block that contains the boundary of the outer face of G . Every separating triangle T of G corresponds to an edge between two 4-blocks, where the parent lies in the exterior of T (plus T) and the child lies in the interior of T (plus T). For a 4-block G_i in \mathcal{B} denote by T_i the outer face of G_i , and denote by n_i the number of vertices of G_i minus three (the vertices of T_i). As a maximal planar graph, G_i has $3(n_i + 3) - 6 = 3(n_i + 1)$ edges and $2(n_i + 3) - 4 = 2(n_i + 1)$ faces. An edge of G_i is called an

interior edge if it is not incident to the outer face T_i . For each 4-block G_i in \mathcal{B} we maintain counters f_i and d_i that denote the number of flips and dummy flips, respectively, that were used within G_i during the course of the algorithm. Initially $f_i = d_i = 0$, for every vertex G_i of \mathcal{B} .

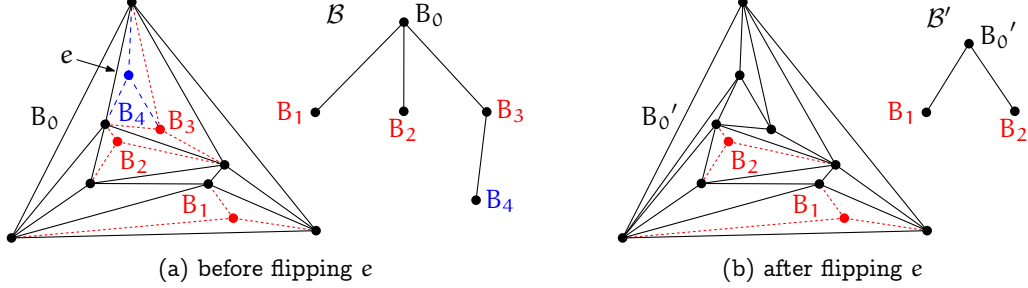


Figure 8: Example of a 4-block decomposition and how a flip of the edge e merges blocks. The vertices and edges of the root (level zero) are shown solid black, the vertices and edges on level one are shown dotted red, and the vertices and edges on level two are shown dashed blue.

The algorithm computes the sequence of flip and dummy flip operations incrementally, and maintains a current triangulation produced by the operations. Both the graph G and the 4-block decomposition \mathcal{B} change dynamically during the algorithm: when we flip an edge e of some separating triangle(s), all 4-blocks containing edge e merge into a single 4-block. At the end of the algorithm, the tree \mathcal{B} consists of a single 4-block that corresponds to the 4-connected graph G' . In order to avoid notational clutter, we always denote the current 4-block tree by \mathcal{B} . As an invariant (detailed below) we maintain that at each node of \mathcal{B} the number of interior edges (ignoring dummy edges) balances the cost of operations that were spent in this 4-block. As \mathcal{B} evolves, so does the graph $\mathcal{G}(\mathcal{B})$ represented by \mathcal{B} . This graph is the union of all nodes (4-blocks) in \mathcal{B} , where for any edge of \mathcal{B} the vertices and edges of the common triangle in the two endpoints (4-blocks) are identified.

Main loop. At every step, we take an arbitrary 4-block G_i on the penultimate level of \mathcal{B} , that is, G_i is not a leaf but all of its children are leaves. Let C_i denote the set of indices c such that G_c is a child of G_i in \mathcal{B} , and denote $\mathcal{T}_i = \{T_c \mid c \in C_i\}$. The algorithm selects a sequence of edges of G_i to be flipped (or dummy flipped) in order to merge G_i with G_c , for all $c \in C_i$, into a new 4-block G_z . Denote the resulting 4-block tree by \mathcal{B}' . If no edge of T_i is flipped, then G_z is a leaf of \mathcal{B}' . But if an edge of T_i is flipped, then G_z may be an interior node of \mathcal{B}' .

Algorithmic preliminaries. In each iteration, we flip the edges of a dual matching of G_i (a 4-connector, defined below), but if \mathcal{T}_i forms a checkerboard (defined below), we substitute three of these flip operations by one dummy flip.

A 4-connector for G_i is a dual matching of G_i that contains precisely one edge from every triangle in \mathcal{T}_i . By Lemma 11 we can flip the edges of a 4-connector in an arbitrary order, and the 4-blocks G_c , for all $c \in C_i$, merge into G_i . Note that a perfect dual matching for G_i consists of $2(n_i + 1)/2 = n_i + 1$ edges and so every 4-connector contains at most this many edges.

Consider a partition of the edge set of G_i into three perfect dual matchings $D_1, D_2,$ and D_3 (Theorem 12). For each $D_i, i \in \{1, 2, 3\}$, the subset M_i of edges that are incident to some triangle from \mathcal{T}_i is a 4-connector for G_i . We select $M \in \{M_1, M_2, M_3\}$ according to the following criteria:

- M has minimum cardinality and
- if possible (among the sets of minimum cardinality), then M contains an edge of T_i .

Every 4-connector that is obtained from some partition D_1, D_2, D_3 in the described way is an *optimal 4-connector* for G_i in \mathcal{B} .

We say that \mathcal{T}_i is a *checkerboard* if every interior edge of G_i belongs to exactly one triangle of \mathcal{T}_i . If \mathcal{T}_i is a checkerboard, then we perform a dummy flip on a triangle F that is selected according to the following lemma (see Figure 9 for illustration).

Lemma 18. *If \mathcal{T}_i is a checkerboard, then G_i contains two triangles, F and H , such that F is a bounded facial triangle adjacent to three triangles in \mathcal{T}_i and H is adjacent to \mathcal{T}_i but not to F .*

Proof. We partition the set of facial triangles of G_i into two subsets: the set \mathcal{T}_i (which are separating triangles in $\mathcal{G}(\mathcal{B})$), and the set of all other faces that we denote by \mathcal{F}_i . The dual graph G_i^* is a 3-regular planar graph on $2n_i + 2$ nodes, one of which corresponds to the outer face T_i .

If \mathcal{T}_i is a checkerboard, then the $2n_i + 1$ bounded faces of G_i induce a bipartite subgraph in G_i^* between \mathcal{T}_i and the bounded faces in \mathcal{F}_i . This subgraph has precisely three vertices of degree two (adjacent to the outer face), all other degrees are three. Since the sum of degrees in the two vertex classes are equal, all three neighbors of the outer face must be in the same vertex class. Therefore, G_i^* is a bipartite graph on all faces, where the two classes are either \mathcal{T}_i and \mathcal{F}_i , or $\mathcal{T}_i \cup \{T_i\}$ and $\mathcal{F}_i \setminus \{T_i\}$. Given that G_i has $2n_i + 2$ faces (including the outer face T_i), the two classes each have size $n_i + 1$.

In particular, T_i is adjacent to three distinct facial triangles of G_i that are either all in \mathcal{T}_i or all in \mathcal{F}_i . We distinguish two cases. First assume T_i is adjacent to three triangles in \mathcal{T}_i . Then \mathcal{F}_i also contains at least three triangles. Since G_i^* is planar, it does not contain $K_{3,3}$ as a subgraph, and so there exists a bounded face $F \in \mathcal{F}_i$ that is not adjacent to all three triangles adjacent to T_i , and a face $H \in \mathcal{T}_i$ adjacent to T_i but not to F . Next assume T_i is adjacent to three triangles in \mathcal{F}_i , let one of them be H . Since these triangles are edge-disjoint, each vertex of T_i is incident to a distinct triangle in \mathcal{T}_i . This implies that $|\mathcal{T}_i \cup \{T_i\}| \geq 4$, and so there is a fourth triangle $F \in \mathcal{F}_i \setminus \{T_i\}$ that is adjacent to three triangles in \mathcal{T}_i . \square

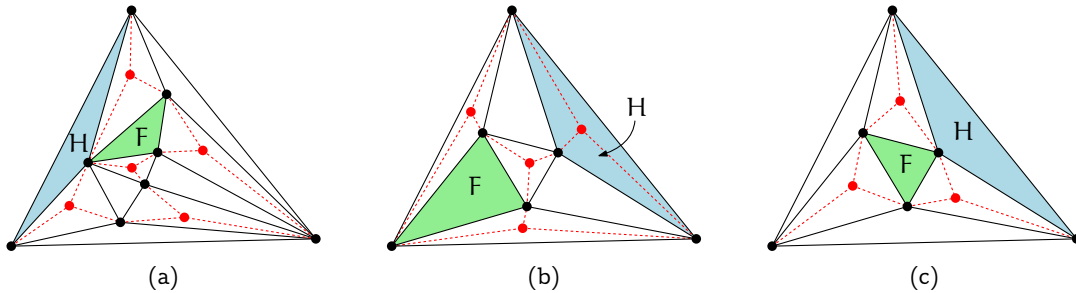


Figure 9: Three examples to illustrate Lemma 18. The vertices and edges of G_i are shown solid black, the vertices and edges of its children are shown dotted red.

Algorithm 4CONNECT(G). Given a triangulation G , fix an arbitrary embedding of G . This embedding defines a rooted 4-block tree \mathcal{B} . While \mathcal{B} is not a singleton, do:

- (1) Consider an arbitrary vertex G_i at the penultimate level of \mathcal{B} .
- (2) If \mathcal{T}_i is *not* a checkerboard, then find an optimal 4-connector M for G_i and flip the edges of M in an arbitrary order.

(3) Otherwise, let F and H be two triangles of G_i as in Lemma 18. Let $D \in \{D_1, D_2, D_3\}$ be the dual perfect matching that contains the common edge of H and T_i . First apply a dummy flip to F . Then consider all triangles in \mathcal{T}_i that are not adjacent to F , in an arbitrary order. For every such triangle, flip the incident edge in D .

(4) Finally, update \mathcal{B} and $\mathcal{G}(\mathcal{B})$.

Correctness of the Algorithm. We show that the above algorithm transforms an input triangulation G on n vertices into a 4-connected triangulation using a sequence of f flips and d dummy flips, for some $f, d \in \mathbb{N}_0$, such that $f + 2d \leq (n - 3)/2$. By Lemmata 11, 16, and 18, the operations described in the algorithm can be performed. In every step of the algorithm at least two nodes of the 4-block tree are merged. Therefore, after a finite number of steps we are left with a block tree that consists of a single 4-block G' .

Independent dummy vertices. The following observation is crucial for the second step of our algorithm (Section 6.2) where we eliminate dummy vertices and simulate dummy flips using regular edge flips.

Observation 19. *For each vertex v created by a dummy flip operation in $4\text{CONNECT}(G)$, subsequent operations do not modify the six facial triangles incident to v .*

Proof. The claim directly follows from the following properties of the operations performed by the algorithm. (i) When the algorithm flips an edge (including the three flips of a dummy flip), this edge is incident to a separating triangle of the current graph. (ii) The algorithm never creates new separating triangles. (iii) For every vertex v created by a dummy flip, at the end of this dummy flip none of the edges of the six triangles incident to v is incident to any separating triangle in $\mathcal{G}(\mathcal{B})$.

The first two properties are obvious, but the third may need a bit of justification: Each of the three edges of the face F where v is inserted is incident to a triangle from \mathcal{T}_i . In particular, the three neighbors of v other than the vertices of F lie inside these triangles and so do all edges between them and the vertices of F . By choice of G_i , the graph inside any triangle from \mathcal{T}_i is a leaf of \mathcal{B} and, therefore, does not contain any separating triangle of $\mathcal{G}(\mathcal{B})$. \square

Free and trapped edges. It remains to bound the number of flip and dummy flip operations performed by the algorithm. An edge within some 4-block G_i of \mathcal{B} is *free* if it is not incident to any separating triangle of $\mathcal{G}(\mathcal{B})$. Free edges are a good measure of progress for our algorithm because our final goal is to arrive at a state where all edges of $\mathcal{G}(\mathcal{B})$ are free. An edge of G_i that is not free is incident to one or two triangles from \mathcal{T}_i . We refer to these edges as *singly trapped* and *doubly trapped*, respectively.

Invariants. As an invariant we maintain that every vertex G_i of \mathcal{B} satisfies the following conditions:

(F1) If G_i is the only vertex of \mathcal{B} , then it has at least $6f_i + 15d_i + 3$ free edges.

(F2) If G_i is a leaf of \mathcal{B} that is not the root of \mathcal{B} , then G_i has at least $6f_i + 15d_i + 3$ free interior edges.

(F3) If G_i is an interior vertex of \mathcal{B} , then either $f_i = d_i = 0$ or G_i has at least $6f_i + 15d_i + 1$ free interior edges.

Initially, (F1) holds since \mathcal{B} has at least two vertices. (F2) holds for every leaf G_i of \mathcal{B} because all of the interior $3(n_i + 1) - 3 = 3n_i$ edges are free, $n_i \geq 1$, and $f_i = d_i = 0$. Finally, (F3) holds for every interior vertex G_i of \mathcal{B} because $f_i = d_i = 0$. Having a certain number of edges in a

plane graph implies having a certain number of vertices, as quantified by the following lemma.

Lemma 20. *If \mathcal{B} has at least two nodes, then $n_i \geq 2f_i + 5d_i + 1$, for every 4-block G_i in \mathcal{B} .*

Proof. For a leaf G_i of \mathcal{B} , condition (F2) implies that G_i has at least $6f_i + 15d_i + 6$ edges (the three edges of T_i are not interior). As G_i has exactly $3(n_i + 1)$ edges, it follows that $n_i \geq 2f_i + 5d_i + 1$. Similarly for an interior vertex G_i of \mathcal{B} with $f_i + d_i > 0$, condition (F3) implies that G_i has at least $6f_i + 15d_i + 4$ edges and so $n_i \geq 2f_i + 5d_i + 1/3$. As n_i is integral, we again obtain $n_i \geq 2f_i + 5d_i + 1$. Finally, if $f_i = d_i = 0$, then the statement becomes $n_i \geq 1$, which is trivial. \square

Invariant maintenance. It remains to show that each step of the algorithm maintains invariants (F1)–(F3). If an edge e of T_i is flipped and G_i is not the root of \mathcal{B} , then more blocks may merge into G_z : The edge e is definitely shared with the parent of G_i in \mathcal{B} , but it may be shared with further ancestors as well. In addition, the edge e may belong to (at most) one sibling G_s of G_i and possibly some descendants of G_s . We denote by J the set of all j such that G_j is a leaf of \mathcal{B} that is merged into G_z . Similarly, denote by Q the set of all q such that G_q is an interior vertex of \mathcal{B} that is merged into G_z , and denote by Q^+ the set of indices $q \in Q$ such that $f_q + d_q > 0$. Note that neither J nor Q are empty, because $C_i \subseteq J$ and $i \in Q$. However, we may have $Q^+ = \emptyset$.

At the end of a step that merged all G_j , for $j \in J \cup Q$, into G_z we have $f_z = f + \sum_{j \in J \cup Q} f_j$ and $d_z = d + \sum_{j \in J \cup Q} d_j$, where f and d denote the number of flips and dummy flips, respectively, that were executed during this step. The following two lemmata do not make specific assumptions about the set of operations (other than that they are valid, that is, yield a triangulation). In particular, the set of edges flipped need not form a *optimal* 4-connector.

Lemma 21. *Suppose that G_i together with all its children in \mathcal{B} is merged into a leaf G_z of \mathcal{B}' using f flips and d dummy flips. Then G_z contains at least $6(f_z - f_i - f) + 15(d_z - d_i - d) + 3n_i + 3|C_i| + 3|Q| - 3$ free interior edges.*

Proof. Combining Lemma 20 with $C_i \subseteq J$ we obtain

$$\begin{aligned} n_z &= \sum_{j \in J \cup Q} n_j \geq n_i + \sum_{j \in (J \cup Q) \setminus \{i\}} (2f_j + 5d_j + 1) \\ &\geq n_i + |J \cup Q| - 1 + 2(f_z - f_i - f) + 5(d_z - d_i - d) \\ &\geq 2(f_z - f_i - f) + 5(d_z - d_i - d) + n_i + |C_i| + |Q| - 1. \end{aligned}$$

Given that G_z is a leaf of \mathcal{B}' , all its $3(n_z + 3) - 9 = 3n_z$ interior edges are free. \square

Lemma 22. *Suppose that G_i along with all its children in \mathcal{B} is merged into an interior node G_z of \mathcal{B}' using f flips and d dummy flips. Then G_z contains at least $6(f_z - f_i - f) + 15(d_z - d_i - d) + 3n_i + 3|C_i| + 1$ free interior edges.*

Proof. All children of G_i are merged together with G_i into G_z . As G_z is an interior node of \mathcal{B}' , an edge of T_i is flipped in this process and the new edge added by this flip is a free interior edge of G_z . In addition, all edges inside T_i are free interior edges of G_z . The number of vertices inside T_i is $n_i + \sum_{j \in C_i} n_j$, which by Lemma 20 is at least $n_i + \sum_{j \in C_i} (2f_j + 5d_j + 1)$. Hence the number of free interior edges inside T_i is at least $3n_i + 3|C_i| + \sum_{j \in C_i} (6f_j + 15d_j)$.

From the remaining nodes of \mathcal{B} merged into G_z we get by (F2) and (F3) an additional number of $\sum_{j \in J \setminus C_i} (6f_j + 15d_j + 3) + \sum_{q \in Q^+} (6f_q + 15d_q + 1)$ free interior edges. Summing up

yields at least

$$\begin{aligned} & 1 + 3n_i + 3|C_i| + \sum_{j \in C_i} (6f_j + 15d_j) + \sum_{j \in J \setminus C_i} (6f_j + 15d_j + 3) + \sum_{q \in Q^+} (6f_q + 15d_q + 1) \\ & = 6(f_z - f_i - f) + 15(d_z - d_i - d) + 3n_i + 3|C_i| + 1 + 3|J \setminus C_i| + |Q^+| \end{aligned}$$

free interior edges in G_z . □

Case analysis. We now show that every step of the algorithm 4CONNECT maintains the invariants (F1)–(F3). We start with the case that \mathcal{T}_i forms a checkerboard and then consider the case that \mathcal{T}_i does not form a checkerboard.

Lemma 23. *Suppose that \mathcal{T}_i is a checkerboard. Then G_z fulfills invariants (F1)–(F3).*

Proof. In this case, the algorithm performs $d = 1$ dummy flip and $f = |C_i| - 3$ flips. As \mathcal{T}_i is a checkerboard, we have $f_i = d_i = 0$ (any previous flip in G_i would have created a free interior edge). Recall that G_i has $2(n_i + 1)$ faces, one of which is the outer face, and hence either $|C_i| = n_i$ or $|C_i| = n_i + 1$ (see also Lemma 18). We distinguish these two cases.

Case 1: $|C_i| = n_i$. Then $f = n_i - 3$. No edge of \mathcal{T}_i is flipped in this step, and G_i along with all its children is merged into a leaf G_z of \mathcal{B}' . By Lemma 21 we find at least

$$6(f_z - (n_i - 3)) + 15(d_z - 1) + 3n_i + 3|C_i| + 3|Q| - 3 = 6f_z + 15d_z + 3|Q|$$

free interior edges in G_z , which noting that $i \in Q$ proves (F2).

Case 2: $|C_i| = n_i + 1$. Then $f = n_i - 2$ and \mathcal{T}_i is adjacent to three distinct triangles from \mathcal{T}_i . By Lemma 18, we have $H \in \mathcal{T}_i$, and H is not adjacent to the triangle F selected for the dummy flip in this step. Consequently, the algorithm flips the common edge of H and \mathcal{T}_i .

If the resulting graph G_z is an interior node of \mathcal{B}' , then by Lemma 22 we find at least

$$6(f_z - (n_i - 2)) + 15(d_z - 1) + 3n_i + 3(n_i + 1) + 1 = 6f_z + 15d_z + 1$$

free interior edges in G_z , which implies (F3). Otherwise, G_z is a leaf of \mathcal{B}' and by Lemma 21 we find at least

$$6(f_z - (n_i - 2)) + 15(d_z - 1) + 3n_i + 3(n_i + 1) + 3|Q| - 3 = 6f_z + 15d_z + 3(|Q| - 1)$$

free interior edges in G_z . If G_z is the only vertex of \mathcal{B}' , then together with the three edges of \mathcal{T}_i and noting that $i \in Q$ we obtain (F1) for G_z . Otherwise, as an edge of \mathcal{T}_i is flipped, also the parent G_p of G_i is merged into G_z . Therefore $\{i, p\} \subseteq Q$ and (F2) holds for G_z . □

The analysis for the case that \mathcal{T}_i does not form a checkerboard is split into two lemmata. Lemma 24 addresses the case that G_i has two separating triangles that share an edge, whereas Lemma 28 discusses the situation that the triangles in \mathcal{T}_i are pairwise edge-disjoint.

Lemma 24. *If G_i contains a doubly trapped edge, then G_z fulfills invariants (F1)–(F3).*

Proof. Let S denote the set of doubly trapped edges in G_i , and put $s = |S|$. As $s \geq 1$, we know that \mathcal{T}_i is not a checkerboard and so the algorithm flips the edges of an optimal 4-connector M . As M contains an edge of every triangle from \mathcal{T}_i , we have $|M| = |C_i| - |M \cap S|$. In particular, the choice of the optimal 4-connector implies $|M \cap S| \geq \lceil s/3 \rceil$.

As we flip only edges that are incident to a separating triangle, all free interior edges of G_j in \mathcal{B} , for $j \in J \cup Q$, remain free interior edges of G_z in \mathcal{B}' . By (F2) and (F3) we obtain the

following lower bound on the number of such edges for $j \in J \cup Q^+$ (but not for $j \in Q \setminus Q^+$, a detail that we will get back to later):

$$\sum_{j \in J} (6f_j + 15d_j + 3) + \sum_{q \in Q^+} (6f_q + 15d_q + 1) = 6(f_z - |M|) + 15d_z + 3|J| + |Q^+|.$$

In addition, all interior edges of G_i that are incident to some triangle in \mathcal{T}_i become free in G_z (some of them may have been flipped). Every triangle in \mathcal{T}_i has three edges, but some of these edges are incident to T_i —denote the number of these edges by $t \in \{0, 1, 2, 3\}$ —or to two triangles of \mathcal{T}_i . Therefore, at least $3|C_i| - s - t$ interior edges of G_i become free and so there are at least

$$\begin{aligned} & 6(f_z - |M|) + 15d_z + 3|J| + |Q^+| + 3|C_i| - s - t \\ & \geq 6f_z + 15d_z + 3(|J| - |C_i|) + |Q^+| + (6\lceil s/3 \rceil - s - t) \end{aligned} \quad (25)$$

$$\geq 6f_z + 15d_z + (6\lceil s/3 \rceil - s - t) \quad (26)$$

free interior edges in G_z , where the first inequality uses $|M| = |C_i| - |M \cap S| \leq |C_i| - \lceil s/3 \rceil$ and the second inequality uses $C_i \subseteq J$. If $|M| \leq |C_i| - \lceil s/3 \rceil - 1$, then the last summand of (26) becomes $6\lceil s/3 \rceil - s - t + 6$. Given that $t \leq 3$, this is at least three and, therefore, the claim follows. Similarly, if $t = 0$, then $6\lceil s/3 \rceil - s - t \geq 3\lceil s/3 \rceil \geq 3$, where the last inequality is due to $s \geq 1$. Again the claim follows. Hence suppose that $t \geq 1$ and $|M| = |C_i| - \lceil s/3 \rceil$. We distinguish two cases.

Case 1: M does not contain an edge of T_i . Let M_1, M_2 , and M_3 denote the three 4-connectors that M was selected from. We need to show that the last summand in (26) is at least three, for which we distinguish three subcases, depending on the residue of $s \bmod 3$.

If $s \equiv 0 \pmod{3}$, then $|M| = |C_i| - s/3$, that is, M contains exactly $s/3$ doubly trapped edges. Every doubly trapped edge appears in exactly one of M_1, M_2 , or M_3 . Therefore $|M| = |M_1| = |M_2| = |M_3| = |C_i| - s/3$. As $t \geq 1$, there is at least one (singly) trapped edge e of T_i . Given that e is trapped, one of M_1, M_2 , or M_3 contains it. Hence, by the definition of optimality, also M contains an edge of T_i , in contradiction to our assumption that it does not.

If $s \equiv 1 \pmod{3}$, then $6\lceil s/3 \rceil - s - t = 6(s+2)/3 - s - t \geq 5 - t$ and so the claim holds unless $t = 3$. If $t = 3$, then all three edges of T_i are (singly) trapped. Therefore, each of M_1, M_2 , and M_3 and, in particular, M contains an edge of T_i , in contradiction to our assumption that it does not.

It remains to consider the case $s \equiv 2 \pmod{3}$, which implies $s \geq 2$. Then $6\lceil s/3 \rceil - s - t = 6(s+1)/3 - s - t \geq s + 2 - t \geq 4 - t$ and so the claim holds unless $t \geq 2$. If $t = 3$, then argue as in the preceding case and arrive at a contradiction. Hence suppose that $t = 2$. Suppose without loss of generality that $M = M_1$. Given that M does not contain an edge of T_i and two edges of T_i are (singly) trapped, both M_2 and M_3 contain an edge of T_i . By the optimality criteria it follows that $|M_2|, |M_3| \geq |M| + 1 = |C_i| - (s+1)/3 + 1 = |C_i| - (s-2)/3$, that is, neither M_2 nor M_3 contains more than $(s-2)/3$ doubly trapped edges. On the other hand, M contains exactly $(s+1)/3$ doubly trapped edges, which leaves $(2s-1)/3$ doubly trapped edges for M_2 and M_3 . But $2(s-2)/3 = (2s-4)/3 < (2s-1)/3$, a contradiction.

Case 2: M contains an edge e of T_i . The last summand in (25) is $6\lceil s/3 \rceil - s - t \geq 3\lceil s/3 \rceil - t \geq 3 - t \geq 0$. In our accounting from (25) none of the edges of T_i is counted as an interior free edge of G_z . But the edge that e is flipped into is a free interior edge of G_z . So we can raise our count by one. Therefore, if G_z is an interior node of \mathcal{B}' , then (F3) holds and the claim follows.

It remains to consider the case that G_z is a leaf of \mathcal{B}' . If the other two edges of T_i (other than e) are both free interior edges of G_z , then the claim follows. Otherwise, at least one edge $g \neq e$ of T_i is not a free interior edge of G_z . As G_z is a leaf of \mathcal{B}' (and so G_z does not have a separating triangle), g is an edge of the outer face T_z of G_z . As any two triangles in a triangulation share

at most one edge, it follows that the third edge of T_i (other than e and g) is a free interior edge of G_z . This increases our count by another edge.

If $Q^+ \neq \emptyset$, then the claim follows. Otherwise, we have $Q^+ = \emptyset$. In particular, for the parent G_p of G_i in \mathcal{B} we have $p \in Q \setminus Q^+$ and so none of the interior edges of G_p have been counted in (25). As every node has at least three interior edges and— G_z being a leaf of \mathcal{B}' —all its interior edges are free—the claim follows. \square

For the case that there are no doubly trapped edges in G_i and some flips or dummy flips have already been executed in G_i , the following lemma provides an upper bound on $|M|$ using the invariants.

Lemma 27. *Let M be an optimal 4-connector for G_i . If the triangles in \mathcal{T}_i are pairwise edge-disjoint but \mathcal{T}_i is not a checkerboard, then $|M| \leq n_i - 2f_i - 5d_i$. Equality is possible only if M contains an edge of T_i .*

Proof. First assume that $f_i + d_i > 0$. Then by (F3), there are at least $6f_i + 15d_i + 1$ free interior edges in G_i . Therefore, at least one of the three perfect dual matchings D_1 , D_2 , or D_3 (Theorem 12), say, D_1 contains at least $\lceil (6f_i + 15d_i + 1)/3 \rceil = 2f_i + 5d_i + 1$ free interior edges. As none of these edges appears in the corresponding 4-connector M_1 , we have $|M| \leq |M_1| \leq (n_i + 1) - (2f_i + 5d_i + 1) = n_i - 2f_i - 5d_i$. In case of equality, M_1 results from removing only free interior edges from a perfect dual matching D_1 . In particular, as the edge of D_1 incident to T_i is not interior, both M_1 and—by definition of optimality— M contain an edge of T_i .

It remains to consider the case $f_i = d_i = 0$. As \mathcal{T}_i is not a checkerboard, G_i has at least one free interior edge. This edge appears in one of the three perfect dual matchings D_1 , D_2 , or D_3 and therefore $|M| \leq n_i = n_i - 2f_i - 5d_i$. If $|M| = n_i$, then suppose contrary to our claim that M does not contain any edge of T_i . Then T_i is not adjacent to any triangle from \mathcal{T}_i . (Otherwise, the corresponding edge e shared by T_i and a triangle from \mathcal{T}_i appears in one of the three 4-connectors M_1 , M_2 , and M_3 that M is selected from. As $|M_1| = |M_2| = |M_3|$, our optimality criterion selects M to be the 4-connector that contains e .) Given that $|M| = n_i$, we conclude that each of the $3n_i$ interior edges of G_i is incident to some triangle from \mathcal{T}_i . But then \mathcal{T}_i is a checkerboard, contrary to our assumption that it is not. \square

Lemma 28. *Suppose that the triangles in \mathcal{T}_i are pairwise edge-disjoint but \mathcal{T}_i is not a checkerboard. Then G_z fulfills invariants (F1)–(F3).*

Proof. By Lemma 27 we have $|M| \leq n_i - 2f_i - 5d_i$. To conclude the analysis we distinguish four cases.

Case 1: G_z is the only node of \mathcal{B}' . Using Lemma 21 with $f = |M|$ and $d = 0$, we find at least

$$6(f_z - f_i - f) + 15(d_z - d_i) + 3n_i + 3|C_i| + 3|Q| - 3 \geq 6f_z + 15d_z + 3|Q| - 3$$

free interior edges in G_z , where the inequality uses $|C_i| = |M| = f$ and $n_i \geq f + 2f_i + 5d_i$. Given that $i \in Q$ and the three outer edges of T_z are not interior edges of G_z , (F1) follows.

Case 2: G_z is an interior vertex of \mathcal{B}' . Then using Lemma 22 with $f = |M|$ and $d = 0$ we find at least

$$\begin{aligned} & 6(f_z - f_i - f) + 15(d_z - d_i) + 3n_i + 3|C_i| + 1 \\ & \geq 6(f_z - f_i) + 15(d_z - d_i) + 3n_i - 3f + 1 \\ & \geq 6f_z + 15d_z + 1 \end{aligned}$$

free interior edges in G_z , where the first inequality uses $|C_i| = |M| = f$ and the second inequality uses $n_i \geq f + 2f_i + 5d_i$. This proves (F3).

Case 3: G_z is a leaf of \mathcal{B}' (but not the only node). We distinguish two subcases. If M does not contain an edge of T_i , then Lemma 27 yields $f = |M| \leq n_i - 2f_i - 5d_i - 1$. By Lemma 21, we find at least

$$\begin{aligned} & 6(f_z - f_i - f) + 15(d_z - d_i) + 3n_i + 3|C_i| + 3|Q| - 3 \\ & \geq 6f_z - 6f_i - 3f - 3(n_i - 2f_i - 5d_i - 1) + 15(d_z - d_i) + 3n_i + 3f + 3|Q| - 3 \\ & = 6f_z + 15d_z + 3|Q| \end{aligned}$$

free interior edges in G_z , where the inequality uses $f \leq n_i - 2f_i - 5d_i - 1$ and $|C_i| = f$. Since $i \in Q$, we have $|Q| \geq 1$ and (F2) follows.

Otherwise, M contains an edge of T_i . Then the parent G_p of G_i in \mathcal{B} is merged into G_z . By Lemma 21 we find at least

$$\begin{aligned} & 6(f_z - f_i - f) + 15(d_z - d_i) + 3n_i + 3|C_i| + 3|Q| - 3 \\ & \geq 6f_z - 6f_i - 3f - 3(n_i - 2f_i - 5d_i) + 15(d_z - d_i) + 3n_i + 3f + 3|Q| - 3 \\ & = 6f_z + 15d_z + 3(|Q| - 1) \end{aligned}$$

free interior edges in G_z , where the inequality uses $f \leq n_i - 2f_i - 5d_i$ and $|C_i| = f$. Since $\{i, p\} \subseteq Q$, we have $|Q| \geq 2$ and (F2) follows. \square

Summary. In all cases we have shown that the resulting 4-block tree \mathcal{B}' satisfies our invariants. Thus the resulting 4-connected graph G' has $n + d$ vertices and at least $6f + 15d + 3$ edges, where f and d denote the number of flip and dummy flip operations, respectively, that were executed during the algorithm. Being a maximal planar graph, G' contains exactly $3(n + d) - 6$ edges. Therefore, $6f + 15d + 3 \leq 3(n + d) - 6$ and so $2f + 4d \leq n - 3$, as required. This completes the proof of Lemma 17.

6.2 Second Step: Eliminate Dummy Vertices

At this stage we have a 4-connected planar graph G' . By Tutte's Theorem such a graph is Hamiltonian, so consider some Hamiltonian cycle H' of G' . It remains to argue how G' and H' can be used to obtain a short sequence of edge flips that transform the original graph G into a Hamiltonian graph G'' . The following lemma in combination with Lemma 17 completes the proof for the first part of Theorem 5 (Hamiltonian graph through flip sequence).

Lemma 29. *Suppose that G' has been obtained from G using f flips and d dummy flips. Then G can be transformed into a Hamiltonian maximal planar graph using at most $f + 2d$ edge flips.*

Proof. While it is obvious how to obtain those edges of $E(G') \setminus E(G)$ that were created using a flip (using exactly this flip), we need to argue a bit more for the vertices inserted by dummy flips.

Consider a dummy vertex v and let $T = abc$ denote the triangle of G in which v has been inserted (cf. Observation 19). Obviously H' uses only two of the six edges incident to v along a path uvw . Our goal is to determine the graph G'' along with a Hamiltonian cycle H'' in G'' . Depending on the relative position of the two edges uv and vw we distinguish three cases.

Case 1: uv and vw are opposite in the circular order of edges incident to v (Figure 10(a)–(b)). Then exactly one of uv or vw crosses an edge, say ab , of T and flipping ab yields an edge uw in G'' . In H'' the edge uw takes the role of the path uvw in H' .

Case 2: uv and vw are adjacent in the circular order of edges incident to v (Figure 10(c)–(d)). Then no flip is needed, because the edge uw is an edge of G already.

Case 3: uv and vw are at distance two in the circular order of edges incident to v (Figure 10(e)–(f)). Then the edges uv and vw each intersect an edge of T , say, uv intersects the edge bc and vw intersects the edge ab of T . Then flipping the edge ab in G yields the edge cw and a subsequent flip of bc yields the desired edge uw .

In every case at most two flips are needed to simulate the sub-path of H' passing through a dummy vertex v using a corresponding edge in G'' . Altogether we obtain a Hamiltonian cycle H'' in G'' corresponding to H' in G' . \square

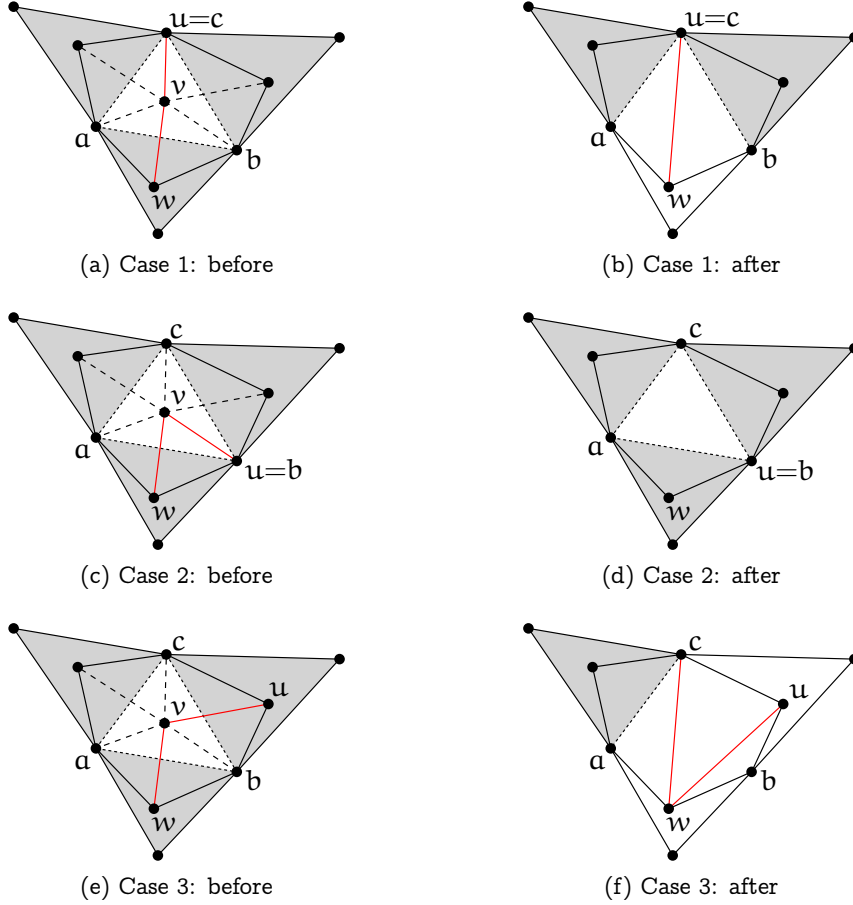


Figure 10: Eliminating a dummy vertex v using edge flips. The six edges incident to v have been added in G' . In this process the three dotted edges of the triangle T in the original graph G have been flipped away. The three separating triangles adjacent to T are shown shaded. The Hamiltonian path H' of G' visits v along the edges shown solid red.

Regarding the second part of Theorem 5 (subhamiltonian graph through edge subdivisions) we make a similar argument by translating both flips and dummy flips into edge subdivisions.

Lemma 30. *Suppose that G' has been obtained from G using f flips and d dummy flips. Then there is a set of at most $f + 2d$ edges in G such that subdividing them results in a subhamiltonian planar graph.*

Proof. Step 1 of our algorithm identified a set S of $f + 3d$ edges in a triangulation G such that each separating triangle is incident to at least one edge in S . The algorithm destroyed all

separating triangles in G by a sequence of f flips and d dummy flips to obtain a 4-connected triangulation G' .

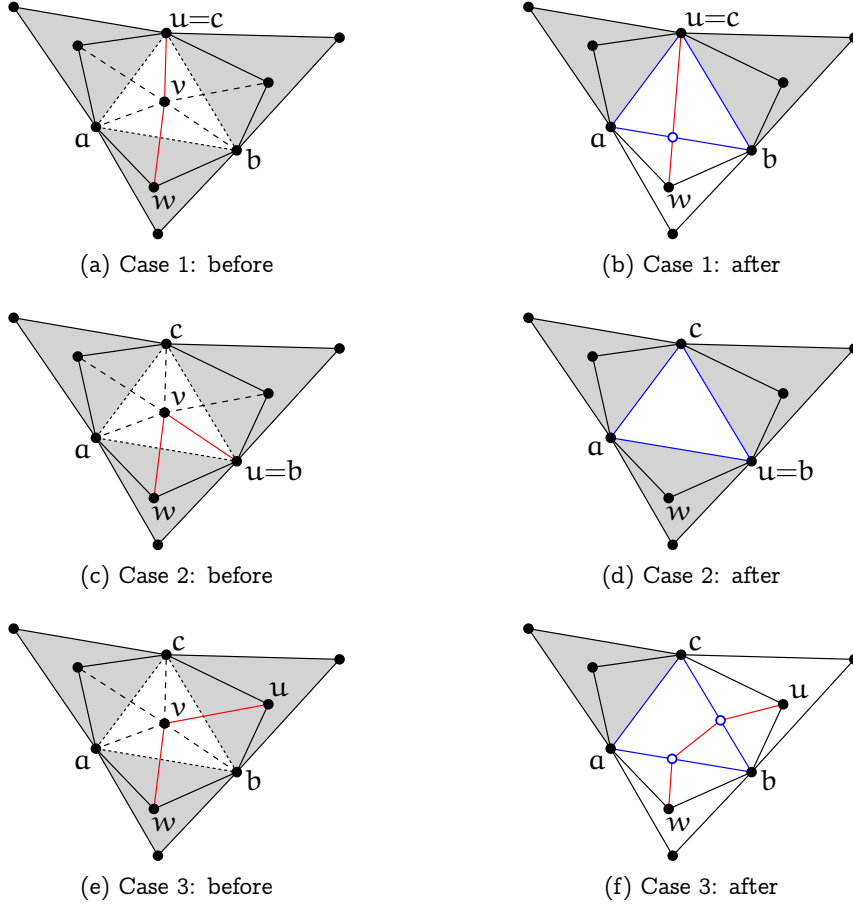


Figure 11: Eliminating a dummy vertex v using edge subdivisions. The six edges incident to v have been added in G^+ . In this process the three dotted edges of the triangle T in the original graph G have been flipped away. The three separating triangles adjacent to T are shown shaded. The Hamiltonian path H^+ of G^+ visits v along the edges shown solid red.

We now destroy all separating triangles of G by a combination of dummy flips and edge subdivisions. Specifically, we construct a graph G^+ from G as follows. Perform all dummy flip operations specified by the algorithm; instead of each flip operation of the algorithm, subdivide the edge with a new vertex; finally insert new edges in the faces incident to subdivision vertices: if a face is incident to precisely one subdivision vertex, then connect it to the opposite vertex of the face; if it is incident to two or three subdivision vertices, then connect all subdivision vertices by an edge or a triangle. Since all separating triangles are destroyed and none created, the resulting graph G^+ is 4-connected. By Tutte's Theorem G^+ contains a Hamiltonian cycle H^+ . It remains to show how to use G^+ and H^+ to replace the d dummy flips with up to $2d$ edge subdivisions.

Consider a dummy vertex v and let $T = abc$ denote the triangle of G in which v has been inserted. By Observation 19 the six triangles incident to v right after its insertion remain untouched throughout the remainder of the algorithm. Clearly, H^+ uses only two of the six edges incident to v along some path uvw . We now determine the graph G^{++} obtained from G with $f + 2d$ subdivision vertices along with a Hamiltonian cycle H^{++} in G^{++} . Similarly to the

proof of Lemma 29, we distinguish three cases depending on the relative position of uv and vw .

Case 1: uv and vw are opposite in the circular order of edges incident to v (Figure 11(a)–(b)). Then exactly one of uv or vw crosses an edge ab of T and subdividing ab yields a new path from u to w in G^{++} . In H^{++} this path takes the role of the path uvw in H^+ .

Case 2: uv and vw are adjacent in the circular order of edges incident to v (Figure 11(c)–(d)). Then no subdivision is needed, because the edge uw is an edge of G already.

Case 3: uv and vw are at distance two in the circular order of edges incident to v (Figure 11(e)–(f)). Then the edges uv and vw each intersect an edge of T , say, uv intersects the edge bc and vw intersects the edge ab of T . Subdividing both ab and bc yields a new path from u to w in G^{++} . In H^{++} this path takes the role of the path uvw in H^+ .

In every case at most two subdivision vertices are needed to simulate the sub-path of H^+ passing through a dummy vertex v using a corresponding path in G^{++} . Consequently, we obtain a Hamiltonian cycle H^{++} in G^{++} corresponding to H^+ in G^+ . \square

Runtime Analysis. It remains to argue that for a maximal planar graph G on n vertices, the algorithm 4CONNECT can be implemented to run in $O(n^2)$ time. The bottleneck is computing an edge-partition into three dual perfect matchings, which is equivalent to vertex 4-coloring and can be done in quadratic time [7, 32]. A 4-block tree \mathcal{B} of the input G can be computed in $O(n)$ time [24]; and we can identify 4-blocks on the penultimate level that are checkerboards in $O(n)$ time. Even though the 4-block tree \mathcal{B} changes in the course of the algorithm, every new 4-block is created by a flip or dummy flip operation, and hence contains a free interior edge. Checkerboards, however, do not contain free interior edges. Consequently, every checkerboard G_i that the algorithm encounters is already present in the input graph at the penultimate level of \mathcal{B} . Therefore it is enough to identify checkerboards during a preprocessing step. It is straightforward to implement Step (3) of 4CONNECT in $O(n)$ time for all checkerboards.

Step (2) considers a node G_i that is not a checkerboard, and requires an optimal 4-connector M , which in turn depends on the edge partition of G_i into three perfect dual matchings. If we call a 4-coloring algorithm for each node G_i in the course of the algorithm, the running time would be $O(n \cdot n^2) = O(n^3)$. However, it suffices to compute an edge partition into three perfect dual matchings $D_1 \cup D_2 \cup D_3$ such that every separating triangle is incident to precisely one edge from each matching (Corollary 14) at preprocessing in $O(n^2)$ time. Since separating triangles are destroyed but never created by the algorithm, each separating triangle is already present in the input graph G , and one edge is contained in each of D_1 , D_2 , and D_3 . Consequently, for each 4-block G_i we can use the restrictions of the initial edge-partition $D_1 \cup D_2 \cup D_3$ to the edges of G_i rather than recomputing a new edge-partition for G_i . Finally, the maintenance of the 4-block tree \mathcal{B} takes $O(n)$ time overall, since every flip and dummy flip merges two nodes of \mathcal{B} and the initial number of nodes is linear.

Our algorithm 4CONNECT returns a 4-connected graph G' with $O(n)$ vertices. A Hamiltonian cycle H' in G' can be computed in $O(n)$ time and space [14]. The local operations in Lemmata 29 and 30 that replace a dummy flip by ordinary flips or subdivisions, respectively, can be implemented in $O(n)$ time.

6.3 Lower bound

The best lower bound we know can be obtained using the following standard construction [2, 22].

Theorem 6. *For every $i \in \mathbb{N}$, there is a maximal planar graph G_i on $n_i = 3i + 8$ vertices such that no sequence of less than $(n_i - 8)/3 = i$ edge flips produces a Hamiltonian graph, and*

there is no set of less than $(n_i - 8)/3 = i$ edges whose subdivision produces a subhamiltonian graph.

Proof. Consider an arbitrary maximal planar graph $G_0 = (V_0, E_0)$ on t vertices, for $t > 4$. Let $G = (V_0 \cup V_1, E)$ be the graph obtained by inserting a vertex in every face of G_0 and connecting it to the vertices bounding the face. G has $n := t + 2t - 4 = 3t - 4$ vertices.

Suppose that there exists a sequence of k flips that transforms G into a Hamiltonian graph H . Let C be a Hamiltonian cycle in H . Since $|V_0| = t$ and $|V_1| = 2t - 4$, there must be at least $2t - 4 - t = t - 4$ vertices in V_1 that are followed by another vertex of V_1 in C . In G , however, V_1 forms an independent set. Note that every flip in G creates at most one edge between vertices of V_1 . Hence, $k \geq t - 4 = (n + 4)/3 - 4 = (n - 8)/3$.

Analogously, suppose that we can obtain a subhamiltonian graph $H_0 = (V_0 \cup V_1 \cup V_2, E')$ by performing $|V_2| = k$ edge subdivisions in G . Augment H_0 to a Hamiltonian graph H and let C be a Hamiltonian cycle in H . Since $|V_0| = t$, $|V_1| = 2t - 4$, and $|V_2| = k$, there must be at least $2t - 4 - t - k = t - k - 4$ vertices in V_1 that are followed by another vertex of V_1 in C . In G and H , however, V_1 forms an independent set. Hence, $k \geq t - 4 = (n + 4)/3 - 4 = (n - 8)/3$. \square

7 Conclusions

We conclude by listing some open problems. An obvious problem is to obtain a tight bound for the maximum number of flips needed to transform any triangulation on n vertices into a Hamiltonian triangulation. We proved an upper bound of roughly $n/2$ (Theorem 5) and a lower bound of roughly $n/3$ (Theorem 6).

We have shown that sometimes fewer flips suffice to reach a Hamiltonian triangulation than a 4-connected one. Is there a similar difference for simultaneous flips? We gave an almost tight bound of roughly $2n/3$ edges to obtain a 4-connected triangulation (Theorem 3). Can we get a better bound on the number of edges in a simultaneous flip when we ask for a Hamiltonian triangulation only? The lower bound to reach a Hamiltonian triangulation is the same as in the case of flip sequences: $n/3$. Again, a gap remains.

Another obvious task is to improve the upper bound for the diameter of the combinatorial flip graph. In the current upper bound, $4n$ of $5n$ flips are spent transforming Hamiltonian triangulations to the canonical one. Perhaps, this is the most promising place to look for improvements. As for lower bounds, a simple degree argument gives a lower bound of roughly $2n$ flips [26]. Recently, Frati [18] improved this bound to roughly $7n/3$ flips.

Finally, there is the algorithmic question of how to efficiently compute a flip sequence between two triangulations that meets the diameter bounds. We gave a quadratic time algorithm whose bottleneck is the computation of a Tait partition (equivalently, a 4-coloring). One perfect dual matching can be computed in linear time [5], but our averaging argument needs an edge partition into perfect dual matchings. Is there a different way to do the accounting that leads to a subquadratic algorithm?

A more ambitious goal is to settle the complexity of the flip distance problem in the combinatorial setting: Given a positive integer k and two triangulations T_1 and T_2 on n vertices, is there a sequence of at most k combinatorial flips that transforms T_1 into T_2 ? In the geometric setting this problem is known to be APX-hard [30].

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